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Quantum Physics of Nanostructures - Problem Set 3

Winter term 2022/2023

Due date: The problem set will be discussed Wednesday, 7.12.2022, 15:15-16:45, Room 114.

6. Quantum Spin Hall Effect

5+5+5 Points

Given the 6-terminal setup shown in Fig. 1, your task will be to determine the Hall resistance R_H and the longitudinal resistance R of a system featuring the so-called quantum spin Hall effect. The charge transport in this particular state of matter, which was realized in specifically designed CdTe/HgTe quantum wells [1, 2], is due to one-dimensional edge states. The edge transport is analogous to the edge transport in the Quantum Hall effect, but occurs *without* an applied external magnetic field.

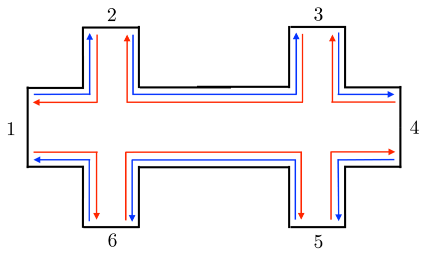

Abbildung 1: In the quantum spin Hall effect, charge transport is due to one-dimensional channels propagating along the sample edges. The spin of the electrons is tied to their motion: left moving (red) and right moving (blue) electrons have opposite spin polarization. The edge channels connect neighboring terminals.

Assume there is no backscattering for the electrons propagating in the one-dimensional edge channels connecting the various voltage probes (terminals). Determine $R_H \equiv R_{14,26} = (V_2 - V_6)/(I_1)$, $R_1 \equiv R_{14,23} = (V_2 - V_3)/(I_1)$ and $R_2 \equiv R_{13,54} = (V_5 - V_4)/(I_1)$ from the Landauer-Büttiker formula for a multi-terminal system,

$$I_i = \sum_j G_{ij} (V_i - V_j) \, ,$$

where $i, j = 1, \ldots, 6$ and I_i denotes the current at terminal i, V_i is the voltage at terminal i, and G_{ij} is the conductance relating current I_i to the voltage difference $V_i - V_j$. The conductances are given by the transmissions T_{ij} from terminal j to terminal i taking into account all relevant edge channels times e^2/h, i.e., $G_{ij} = \frac{e^2}{h} T_{ij}$. Proceed as follows:

(a) Determine the conductance matrix G_{ij} in the absence of backscattering. Take $G_{ij} = 0$ if terminals i and j are not connected by an edge channel according to Fig. 1.

(b) Find the coefficient matrix M_{ij} that relates the current vector I_i to the voltage vector V_i, $I_i = \sum_j M_{ij} V_j$.

(c) Solve for the voltages, assuming that $I_2 = I_3 = 0$ and $I_5 = I_6 = 0$, as appropriate for measuring the voltage drops $V_2 - V_3$, and $V_2 - V_6$, and determine R_H and R_1. Modify the above computation appropriately to determine R_2.

Hint: Note that the equations specified by M_{ij} are not independent due to Kirchhoff rules. Use the freedom to fix e.g. $V_4 = 0$ to drop the fourth row and column from M_{ij} to arrive at an invertible 5×5 coefficient matrix.

[1] M. König *et al.*, Science **318** 5851, 766-770 (2007).

[2] A. Roth *et al.*, Science **325** 5938, 294-297 (2009).

7. Byers-Yang theorem and gauge transformation *2 + 3 + 3 + 2 Points*

The Byers-Yang theorem states that all physical properties of a doubly connected system enclosing a magnetic flux Φ are periodic in the flux with period of flux quantum $\Phi_0 = h/e$. Consider the setup illustrated in the figure below, where an electron moves around an impenetrable cylinder(solenoid) with radius a. The magnetic field inside the cylinder is $\boldsymbol{B} = B\hat{z}$ with total flux Φ, while the electron moves in a region with $\boldsymbol{B} = \nabla \times \boldsymbol{A} = 0$.

(a) The magnetic vector potential outside the cylinder may be obtained using a line integral of the vector potential along a closed contour

$$\Phi = \oint_C \boldsymbol{A} \cdot d\boldsymbol{l}$$

with the line element $d\boldsymbol{l} = dr\hat{r} + rd\phi\hat{\phi} + dz\hat{z}$ in cylindrical coordintes. By using a circle as contour, show that the vector potential in the region outside the cylinder is given by $\boldsymbol{A} = \Phi/2\pi r\,\hat{\phi}$.

(b) The time-independent Schrödinger equation of an electron moving in a region with vector potential $\boldsymbol{A}$ is given by $\boldsymbol{H}\psi = E\psi$ with Hamiltonian

$$H \;=\; \left[\frac{1}{2m}(\boldsymbol{p} - e\boldsymbol{A})^2\right]\,,$$

where $\boldsymbol{p} = -i\hbar\nabla$ is the canonical momentum. Apply the gauge transformation

$$\boldsymbol{A} \to \boldsymbol{A}' = \boldsymbol{A} + \nabla\chi\,, \quad \psi \to \psi' = \exp(i\frac{e}{\hbar}\chi)\psi$$

and show that $\boldsymbol{H}'\psi' = E\psi'$ is indeed equivalent to $\boldsymbol{H}\psi = E\psi$.

(c) Using the explicit form of the vector potential given above, construct a gauge transformation such that $\nabla\chi = -\boldsymbol{A}$. What is the benefit of this transformation? For which values of the flux Φ is the wave function ψ' single-valued?

(d) Find a general expression for a gauge transformation χ which removes the effect of a given vector potential, and study under which conditions the new wave function is single valued. Using such a gauge transformation, provide an argument for the validity of the Byers-Yang theorem.